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1999 J. Phys. A: Math. Gen. 32 5679

(http://iopscience.iop.org/0305-4470/32/30/312)

J. Phys. A: Math. Gen. **32** (1999) 5679–5691. Printed in the UK PII: S0305-4470(99)99843-8

The ground state energy of a massive scalar field in the background of a semi-transparent spherical shell

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Received 30 November 1998, in final form 30 April 1999

Abstract. We calculate the zero-point energy of a massive scalar field in the background of an infinitely thin spherical shell given by a potential of the delta-function type. We use zeta-functional regularization and express the regularized ground state energy (GSE) in terms of the Jost function of the related scattering problem. We then find the corresponding heat-kernel coefficients and perform the renormalization, imposing the normalization condition that the GSE vanishes when the mass of the quantum field becomes large. Finally, the GSE is calculated numerically. Corresponding plots are given for different values of the strength of the background potential, for both attractive and repulsive potentials. The formal transition from a delta-function potential to Dirichlet boundary conditions is not found to take place in the renormalized GSE.

1. Introduction

In recent years much interest has turned to the study of the Casimir energy for spherical configurations. The issue concerns in particular the attempt to explain, by means of quantum field theory (QFT), the puzzling phenomenon of sonoluminescence [1], that is the emission of short intense pulses of light by collapsing bubbles of air in water. Up to now the hypothesis that Casimir energy could play a role in the photon emission has not been supported enough by a satisfactory QFT model for the dielectric *ball*. In this context many authors have studied a spherical shell in the vacuum of the electromagnetic field. Pioneering work on this configuration is found in [2]. Also, in the last two decades, the bag model [3] has generated much interest in spherical configurations. In papers [4, 7] the ground state energy (GSE) of massive fields in the background of a perfectly reflecting shell (a bag) was investigated. A natural extension of this problem is to consider boundary conditions which become transparent at high frequencies, what one would expect for physical reasons. One of the simplest models is a semi-transparent spherical shell realized by a delta-function potential (equation (3) below). It is also interesting as some kind of intermediate configuration between 'hard' boundary conditions and smooth background potentials. As was discussed in [5], the delta-function potential has some features in common with the dielectric background.

In this paper the GSE of a massive scalar field in the background of a spherically symmetric delta-function potential is calculated, analytically as far as possible and numerically in the remaining part. The technique developed in [10] for smooth background potentials is used. It turns out to be well suited for the delta-function potential, which is of course also not smooth. This technique makes use of the zeta-functional regularization, which is well known [8, 9].

0305-4470/99/305679+13$30.00

The heat-kernel coefficients whose knowledge is required for the renormalization, were first derived in [5]. Here they are rederived in the course of calculations. The calculation of the GSE is based on the knowledge of the Jost function of the associated scattering problem, which is given by a simple formula in terms of Bessel functions.

In the next section the general setup of the renormalization procedure is discussed. In section 3 the necessary tools from the scattering theory and the general formulae for the GSE are collected. In section 4 these formulae are specified for the delta-function potential, the renormalization is carried out and the analytic part of the GSE is calculated. Section 5 contains the investigation of the asymptotic behaviour and in section 6 the numerical results for the remaining part are presented. The results are discussed in the conclusions.

2. The model

We want to study the GSE of the scalar field $\varphi(t,\vec{x})$ (to be quantized) in the background of a potential $V(r)$. We consider the following field equation:

$$(\Box + m^2 + V(r))\varphi(x) = 0 \tag{1}$$

where m is the mass of the field. The spherical shell is a geometrical object with radius R and a surface S, to whom a classical energy in terms of classical parameters can be associated. The total energy of the system reads

$$\begin{aligned} E_{TOT} &= E_{class} + E_{quant} \\ &= \left(pV + \sigma S + FR + k + \frac{h}{R}\right) + \left(\frac{1}{2}\sum_n \omega_n\right) \end{aligned} \tag{2}$$

where V is the volume of the sphere, p is the pressure, σ is the surface tension and F, k and h are other parameters with no special names. The classical part of the energy is expressed in a general form in which the dependence on powers of R is explicit. This definition is suitable (as we discuss later) for its renormalization, it was introduced in [6] and used in many works concerning the bag model and the Casimir energy for fermionic and scalar fields with spherical boundaries [7]. The quantum contribution in (2) is the traditional expression for the vacuum energy of a scalar field whose energy eigenvalues are ω_n. To render the eigenvalues of the energy discrete we temporarily take a finite quantization volume with radius $L \gg 1$.

The classical shell is static and spherically symmetric. It is described by a potential

$$V(r) = \frac{\alpha}{R}\delta(r - R) \tag{3}$$

where α is the strength of the potential. The semi-transparency of the boundary will be discussed later. The potential could also be expressed in other forms involving the mass, for instance as

$$V(r) = \alpha m \delta(r - R)$$

since both m and R are dimensional parameters. However, the choice of R is the most natural since the mass concerns the quantum field while the radius concerns the background potential, which is independent from the field.

The quantum contribution to the total energy is divergent, for the regularization we adopt the zeta-function technique. (For a review about zeta-regularization techniques see, e.g., [8,9].) We define the regularized GSE as

$$E_\varphi = \tfrac{1}{2}\sum_{(n)}(\lambda_{(n)}^2 + m^2)^{1/2-s}\mu^{2s} \tag{4}$$

where μ is an arbitrary mass parameter, s is the regularization parameter which we put to zero after renormalization and $\lambda_{(n)}$ are the eigenvalues of the wave equation

$$[-\Delta + V(x)]\phi_{(n)}(x) = \lambda_{(n)}^2 \phi_{(n)}(x). \tag{5}$$

Now we introduce a zeta function. The zeta function of the wave operator with potential $V(r)$ as defined in (5) is

$$\zeta_V(s) = \sum_{(n)} (\lambda_{(n)}^2 + m^2)^{-s}. \tag{6}$$

We can express the GSE in terms of this zeta function:

$$E_\varphi = \tfrac{1}{2}\zeta_V(s - \tfrac{1}{2})\mu^{2s}. \tag{7}$$

Using

$$x^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \mathrm{e}^{-xt} \tag{8}$$

we can write equation (6) in the following form:

$$\zeta_V(s) = \sum_{(n)} \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \mathrm{e}^{-\lambda_{(n)}^2 t - m^2 t} \tag{9}$$

or

$$\zeta_V(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \mathrm{e}^{-m^2 t} \sum_{(n)} \mathrm{e}^{-\lambda_{(n)}^2 t} \tag{10}$$

that is

$$\zeta_V(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \mathrm{e}^{-m^2 t} K(t). \tag{11}$$

The function $K(t)$ is the heat kernel. Taking its asymptotic expansion for $t \to 0$,

$$K(t) = \sum_{(n)} \exp(-\lambda_{(n)}^2 t) \overset{t\to 0}{\sim} \left(\frac{1}{4\pi t}\right)^{3/2} \sum_{j=0}^{\infty} A_j t^j \qquad j = 0, \tfrac{1}{2}, 1, \ldots \tag{12}$$

and making the substitution $s \to s - \frac{1}{2}$ in equation (11), we get an expansion of E_φ in which it is easy to isolate all the divergent (pole) terms. This makes it possible to define the total divergent contribution to the GSE by

$$E_\varphi^{div} = -\frac{m^4}{64\pi^4}\left(\frac{1}{s} + \ln\frac{4\mu^2}{m^2} - \frac{1}{2}\right)A_0 - \frac{m^3}{24\pi^{3/2}}A_{1/2} + \frac{m^2}{32\pi^4}\left(\frac{1}{s} + \ln\frac{4\mu^2}{m^2} - 1\right)A_1$$
$$+\frac{m}{16\pi^{3/2}}A_{3/2} - \frac{1}{32\pi^2}\left(\frac{1}{s} + \ln\frac{4\mu^2}{m^2} - 2\right)A_2. \tag{13}$$

The quantities A_j are the heat-kernel coefficients. In definition (13) we have included terms $A_{1/2}$ and $A_{3/2}$ with a half-integer index, which do not contain poles, to satisfy a normalization condition (see below). The terms containing poles of the form $1/s$ contribute to the ultraviolet divergencies. In a generic smooth background potential $V_g(x)$ their corresponding cofficients are well known:

$$\begin{aligned} A_0 &= \int \mathrm{d}^3 x \\ A_1 &= -\int \mathrm{d}^3\, V_g(x) \\ A_2 &= \tfrac{1}{2}\int \mathrm{d}^3 x\, V_g^2(x). \end{aligned} \tag{14}$$

For the delta potential $V(r)$ the first two equations still hold, while the coefficient A_2 cannot be obtained from this formula because it would contain a squared delta function and must be calculated in a different way. We define the renormalized zero-point energy by

$$E_\varphi^{ren} = E_\varphi - E_\varphi^{div}. \tag{15}$$

To keep the total energy of the system unchanged we add the subtracted object E_φ^{div} to the classical energy. Then we also have a definition of a new classical energy

$$\epsilon_{class} = E_{class} + E_\varphi^{div}. \tag{16}$$

The transition from E_{class} to ϵ_{class} consists in the renormalization of the classical parameters contained in equation (2). Since the heat-kernel coefficients are geometrical coefficients depending on the background (and in our case containing powers of R), each term in the classical energy will dimensionally correspond to a term in E_φ^{div}, then we renormalize as follows:

$$\begin{aligned}
&pV \rightarrow pV - \frac{m^4}{64\pi^4}\left(\frac{1}{s} + \ln\frac{4\mu^2}{m^2} - \frac{1}{2}\right)A_0\\
&\sigma S \rightarrow \sigma S - \frac{m^3}{24\pi^{3/2}}A_{1/2}\\
&FR \rightarrow FR + \frac{m^2}{32\pi^4}\left(\frac{1}{s} + \ln\frac{4\mu^2}{m^2} - 1\right)A_1\\
&k \rightarrow k + \frac{m}{16\pi^{3/2}}A_{3/2}\\
&h/R \rightarrow h/R - \frac{1}{32\pi^2}\left(\frac{1}{s} + \ln\frac{4\mu^2}{m^2} - 2\right)A_2.
\end{aligned} \tag{17}$$

However, in our particular case, both the A_0 and $A_{1/2}$ coefficients, corresponding respectively to p and σ, will turn out to be zero†, then only the last three terms in (17) will undergo renormalization. Now we have

$$E_{TOT} = \epsilon_{class} + E_\varphi^{ren}.$$

The old classical energy E_{class} as defined in (2) is an infinite quantity (i.e. unphysical) since experimentally we can only observe an energy which includes the vacuum fluctuations. The term h/R in (17) deserves a particular attention. In fact, in the case of a massless quantum field the vacuum energy takes the form $\sim 1/R$. Therefore, the classical and quantum contributions would not be distinguishable and the calculation of E_φ^{ren} would lose its predictive power. This difficulty makes it impossible (as expressed in section 2 of paper [5]) to apply our procedure to the case $m = 0$. Furthermore, we must note that the GSE proposed in (15) does not yet have a unique meaning. For the uniqueness of E_φ^{ren} we impose the normalization condition

$$\lim_{m\rightarrow\infty} E_\varphi^{ren} = 0 \tag{18}$$

which physically means that for a field of infinite mass we have no quantum fluctuations. We fulfil this requirement by subtracting all the contributions in E_φ^{div} proportional to the non-negative powers of the mass. That is, we also subtract terms with fractionary indices up to and including the term resulting from the heat-kernel coefficient A_2. The remaining part, containing only negative powers of m, will go to zero for $m \rightarrow \infty$. Note that condition (18) does not apply to a massless field.

† More exactly, the contribution of A_0 does not depend on the background and can be simply ignored.

3. Representation of the GSE in terms of the Jost function

We adopt the approach which appeared for the first time in [10] for the calculation of the GSE in the background of a smooth potential. The method for the calculation of the heat-kernel coefficients for different boundary conditions was developed in an earlier work [11].

In the background of a spherical potential we have a radial Schrödinger equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{l(l+1)}{r^2} - V(r) + \lambda_{n,l}^2\right)\phi_{n,l}(r) = 0 \tag{19}$$

where l is the angular momentum. In the general scattering theory with a continuous spectrum p we have the 'regular solution' [13] defined as

$$\phi_{p,l}(r) \overset{r\to 0}{\sim} j_l(pr) \tag{20}$$

where $j_l(pr)$ is the Riccati–Bessel function. The asymptotics of the regular solution are expressed in terms of the Jost function $f_l(p)$

$$\phi_{p,l}(r) \overset{r\to\infty}{\sim} \frac{\mathrm{i}}{2}(f_l(p)\hat{h}_l^-(pr) - f_l^\star(p)\hat{h}_l^+(pr)) \tag{21}$$

where $\hat{h}_l^\pm(pr)$ are the Riccati–Hankel functions. Now we examine the field at the boundary of our quantization volume. As the potential has a compact support then, at the boundary, expression (21) becomes an exact equation. It can be considered as an equation for the eigenvalues $p = \lambda_{n,l}$. Now taking for instance Dirichlet boundary conditions: $\phi_{p,l}(L) = 0$, we have

$$(f_l(p)\hat{h}_l^-(pL) - f_l^\star(p)\hat{h}_l^+(pL)) = 0. \tag{22}$$

Now, since equation (22) is satisfied for $p = \lambda_{l,n}$, we can rewrite the sum in (4) as a contour integral using the Cauchy theorem

$$E_\varphi = \mu^{2s}\sum_{l=0}^{\infty}(l+1/2)\int_\gamma \frac{\mathrm{d}p}{2\pi\mathrm{i}}(p^2+m^2)^{1/2-s}\frac{\partial}{\partial p}\ln(f_l(p)\hat{h}_l^-(pL) - f_l^\star(p)\hat{h}_l^+(pL)) \tag{23}$$

where the contour γ encloses all the solutions of equation (22) on the positive real p-axis and also the bound state solutions in the limit $L\to\infty$, which lie on the imaginary axis. We further simplfy equation (23) by separating the contour into two pieces γ_1 and γ_2, and expanding the Hankel functions for large L. Then it is possible to recognize in the integrand a term $\mathrm{i}pL$ which corresponds to the Minkowski space contribution. This term can be dropped. Now we shift the two contours γ_1 and γ_2 to the imaginary axis and substitute $p\to\mathrm{i}k$. We then obtain

$$E_\varphi = -\frac{\cos\pi s}{\pi}\mu^{2s}\sum_{l=0}^{\infty}(l+1/2)\int_m^\infty \mathrm{d}k\,[k^2-m^2]^{1/2-s}\frac{\partial}{\partial k}\ln f_l(\mathrm{i}k). \tag{24}$$

Since at the end our quantization volume will go to infinity ($L\to\infty$) this equation will be independent from the boundary condition chosen for the quantization volume. Equation (24) is a very general and useful representation of the GSE, where all the information about the background potential is contained in $f_l(ik)$, and possible bound states as well.

In order to perform the analytical continuation to $s = 0$ and the subtraction proposed in (15) we split E_φ^{ren} into two suitable parts, one of which is divergence free. We obtain this by adding and subtracting the uniform asymptotic expansion of the Jost unction (for more details on this procedure see [12]). We define

$$E_\varphi^{ren} = E_f + E_{as} \tag{25}$$

$$E_f = -\frac{\cos\pi s}{\pi}\mu^{2s}\sum_l\left(l+\frac{1}{2}\right)\int_m^\infty \mathrm{d}k\,[k^2-m^2]^{1/2-s}\frac{\partial}{\partial k}[\ln f_l(\mathrm{i}k) - \ln f_l^{as}(\mathrm{i}k)] \tag{26}$$

and

$$E_{as} = -\frac{\cos \pi s}{\pi} \mu^{2s} \sum_l \left(l + \frac{1}{2}\right) \int_m^\infty \mathrm{d}k\, [k^2 - m^2]^{1/2-s} \frac{\partial}{\partial k} \ln f_l^{as}(\mathrm{i}k) - E_\varphi^{div} \tag{27}$$

where $f_l^{as}(\mathrm{i}k)$ is the uniform asymptotic expansion of the Jost function which we will take up to the third order in $\nu \equiv l + \frac{1}{2} \to \infty$. In fact, three orders are sufficient to make (26) converge and allow to put $s = 0$ under the sign of the sum and the integral. Higher orders could be included to speed up the convergence, but as the reader can check, the final result, that is the quantity E_φ^{ren} introduced in (15), remains unaltered in whatever order (>3) in ν ln $f_l^{as}(\mathrm{i}k)$ has been taken because the asymptotics subtracted in the integrand of (26) are added again in (25).

Now we need the Jost function corresponding to our scattering problem, so we turn to study the background potential.

4. Jost function of the δ-shell

The initial field equation

$$(\Box + m^2 + V(r))\varphi(r) = 0 \tag{28}$$

valid for $-\infty < r < \infty$ can be divided into two parts: an equation for the free field

$$\text{at} \quad r \neq R \longrightarrow (\Box + m^2)\varphi(r) = 0 \tag{29}$$

and an equation for the field on the shell, which includes the required transparency conditions

$$\text{at} \quad r = R \begin{cases} \phi'(R+0) - \phi'(R-0) = \dfrac{\alpha}{R}\phi(R) \\ \phi \text{ being continuous.} \end{cases} \tag{30}$$

For the delta potential the regular solution is

$$\phi_{k,l}(r) = j_l(kR)\Theta(R-r) + \frac{\mathrm{i}}{2}(f_l(k)\hat{h}_l^-(kR) - f_l^\star(k)\hat{h}_l^+(kR))\Theta(r-R) \tag{31}$$

consisiting of two pieces inside and outside the radius R, respectively. As above, $j_l(kR)$ is the Riccati–Bessel function and $h_l^\pm(kR)$ are the Riccati–Hankel functions. Combining equation (30) with (31) we get

$$\begin{aligned} j_l(kR) &= \frac{\mathrm{i}}{2}(f_l(k)\hat{h}_l^-(kR) - f_l^\star(k)\hat{h}_l^+(kR)) \\ \frac{\alpha}{R} j_l(kR) &= k\left(\frac{\mathrm{i}}{2}(f_l(k)\hat{h}_l'^-(kR) - f_l^\star(k)\hat{h}_l'^+(kR)) - j_l'(kR)\right). \end{aligned} \tag{32}$$

We solve for $f_l(k)$, keeping in mind that the Wronskian determinant of $\hat{h}_l^\pm$ is 2i. We find

$$f_l(k) = \frac{1}{2\mathrm{i}}\left(-2\mathrm{i}(-1) + 2\mathrm{i}\frac{\alpha}{kR} j_l(kR)\hat{h}_l^+(kR)\right) \tag{33}$$

or

$$f_l(k) = 1 + \frac{\alpha}{kR} j_l(kR)\hat{h}_l^+(kR). \tag{34}$$

For the Jost function on the imaginary axis we get

$$f_\nu(\mathrm{i}k) = 1 + \alpha I_\nu(kR)K_\nu(kR) \tag{35}$$

which is in terms of the modified Bessel functions, I_ν and K_ν, where $\nu = l + \frac{1}{2}$. We also need the asymptotics of the Jost function: $f_\nu^{as}(\mathrm{i}k)$, or more exactly the logarithms of $f_\nu^{as}(\mathrm{i}k)$. The expansion of the product of the two Bessel functions in (35), for k and ν equally large, is easily obtained with the help of [14]. Then we find the needed asymptotics as a sum of negative powers of ν with coefficients $X_{j,n}$ depending on α. We define

$$\begin{aligned}\ln f_\nu^{as}(\mathrm{i}k) &\equiv \sum_{n=1}^{3}\sum_j X_{j,n}\frac{t^j}{\nu^n}\\ &= \frac{\alpha}{2}\frac{t}{\nu} - \frac{\alpha^2}{8}\frac{t^2}{\nu^2} + \frac{\alpha}{16}\frac{t^3}{\nu^3} + \frac{\alpha^3}{24}\frac{t^3}{\nu^3} - \frac{3\alpha}{8}\frac{t^5}{\nu^3} + \frac{5\alpha}{16}\frac{t^7}{\nu^3}\end{aligned} \tag{36}$$

with $t = 1/\sqrt{1+\frac{k^2R^2}{\nu^2}}$. Now, inserting (35) and (36) in (26) and (27) the renormalized GSE $E_\varphi^{ren} = E_f + E_{as}$ can be calculated. Let us first begin with an analytical simplification of E_{as}. We transform the sum over l into an integral with the help of the known Abel–Plana formula [15]

$$\sum_{l=0}^{\infty} F\left(l+\frac{1}{2}\right) = \int_0^\infty \mathrm{d}\nu\, F(\nu) + \int_0^\infty \frac{\mathrm{d}\nu}{1+\mathrm{e}^{2\pi\nu}}\frac{F(\mathrm{i}\nu)-F(-\mathrm{i}\nu)}{\mathrm{i}}. \tag{37}$$

In our case we have

$$F(\nu) = \int_m^\infty \mathrm{d}k\, \nu[k^2-m^2]^{1/2-s}\frac{\partial}{\partial k}\ln f_\nu^{as}(\mathrm{i}k) \tag{38}$$

which satisfies the validity conditions for equation (37). E_{as} is split into two addends:

$$E_{as}^{(1)} = -\frac{\cos\pi s}{\pi}\mu^{2s}\int_0^\infty \mathrm{d}\nu\, F(\nu) \tag{39}$$

and

$$E_{as}^{(2)} = -\frac{\cos\pi s}{\pi}\mu^{2s}\int_0^\infty \frac{\mathrm{d}\nu}{1+\mathrm{e}^{2\pi\nu}}\frac{(F(\mathrm{i}\nu)-F(-\mathrm{i}\nu))}{\mathrm{i}}. \tag{40}$$

First we calculate $E_{as}^{(1)}$; for the k- and ν-integrations we use the formula

$$\int_0^\infty \mathrm{d}\nu\,\nu\int_m^\infty \mathrm{d}k\,[k^2-m]^{1/2-s}\frac{\partial}{\partial k}\frac{t^j}{\nu^n} = -m^{1-2s}\frac{\Gamma(\frac{3}{2}-s)\Gamma(1+\frac{j-n}{2})\Gamma(s+\frac{n-3}{2})}{2(mR)^{n-2}\Gamma(\frac{j}{2})} \tag{41}$$

then

$$\begin{aligned}E_{as}^{(1)} &= -\frac{\cos\pi s}{\pi}\mu^{2s}\int_0^\infty \mathrm{d}\nu\,\nu\int_m^\infty \mathrm{d}k\,[k^2-m]^{1/2-s}\frac{\partial}{\partial k}\ln f_\nu^{as}(\mathrm{i}k))\\ &= \left(\frac{m^{1-2s}\mu^{2s}}{\pi}\right)\sum_{j,n}X_{j,n}\frac{(mR)^{2-n}}{2}\frac{\Gamma(\frac{3}{2}-s)\Gamma(1+\frac{j-n}{2})\Gamma(s+\frac{n-3}{2})}{\Gamma(\frac{j}{2})}.\end{aligned} \tag{42}$$

Here, inserting the coefficients of (36) and expanding up to the first order in s all the terms which depend on the renormalization parameter we get

$$E_{as}^{(1)} = \frac{2\alpha^3-\alpha}{96\pi R}\left(\frac{1}{s}+\ln\frac{4\mu^2}{m^2}-2\right) - \frac{Rm^2\alpha}{8\pi}\left(\frac{1}{s}+\ln\frac{4\mu^2}{m^2}-1\right)+\frac{m\alpha^2}{16}.$$

The terms proportional to m^2 and m^0 contribute to the divergency of the energy. They are used to calculate the heat-kernel coefficients in (13) and will disappear after the subtraction of E_φ^{div}. The term proportional to m corresponds to the $A_{3/2}$ term of the heat-kernel expansion; although this term generates no divergency it will be as well subtracted as mentioned in section 2, because of our normalization condition.

Now we calculate $E_{as}^{(2)}$. The integration over k is carried out with the formula

$$\int_m^\infty \mathrm{d}k\,[k^2-m]^{1/2-s}\frac{\partial}{\partial k}t^j = -m^{1-2s}\frac{\Gamma(\frac{3}{2}-s)\Gamma(s+\frac{j-1}{2})}{\Gamma(\frac{j}{2})}\frac{(\frac{\nu}{mR})^j}{(1+(\frac{\nu}{mR})^2)^{s+\frac{j-1}{2}}}. \tag{43}$$

We obtain

$$E_{as}^{(2)} = \frac{\cos\pi s}{\pi}\mu^{2s}m^{1-2s}\sum_{j,n}X_{j,n}\frac{\Gamma(\frac{3}{2}-s)\Gamma(s+\frac{j-1}{2})}{\Gamma(\frac{j}{2})}\frac{1}{(Rm)^j}$$
$$\times\int_0^\infty\frac{\mathrm{d}\nu\,\nu}{1+\mathrm{e}^{2\pi\nu}}\left(\frac{(\mathrm{i}\nu)^{j-n}}{(1+(\frac{\mathrm{i}\nu}{mR})^2)^{s+\frac{j-1}{2}}}+\frac{(-\mathrm{i}\nu)^{j-n}}{(1+(\frac{-\mathrm{i}\nu}{mR})^2)^{s+\frac{j-1}{2}}}\right). \qquad (44)$$

This expression can be transformed into

$$E_{as}^{(2)} = -\frac{1}{32\pi^2}\left(\frac{1}{s}+\ln\frac{4\mu^2}{m^2}-2\right)-\frac{\alpha}{2\pi R}\,h_1(Rm)-\frac{\alpha^2}{8R^2m}h_2(Rm)$$
$$+\frac{2\alpha^3+3\alpha}{48\pi R}h_3(Rm)-\frac{\alpha}{8\pi R}\,h_4(Rm)+\frac{\alpha}{48\pi R}h_5(Rm) \qquad (45)$$

where integration by parts was used. The following functions containing the ν-integrations in equation (44) are introduced:

$$\begin{aligned}
h_1(x) &= \int_0^\infty \mathrm{d}\nu\,\frac{\nu}{1+\mathrm{e}^{2\pi\nu}}\ln\left|1-\frac{\nu^2}{x^2}\right|\\
h_2(x) &= \int_0^x \mathrm{d}\nu\,\frac{\nu}{1+\mathrm{e}^{2\pi\nu}}\frac{1}{\sqrt{1-\frac{\nu^2}{x^2}}}\\
h_3(x) &= \int_0^\infty \mathrm{d}\nu\,\ln\left|1-\frac{\nu^2}{x^2}\right|\left(\frac{\nu^2}{1+\mathrm{e}^{2\pi\nu}}\right)'\\
h_4(x) &= \int_0^\infty \mathrm{d}\nu\,\ln\left|1-\frac{\nu^2}{x^2}\right|\left(\frac{1}{\nu}\left(\frac{\nu^2}{1+\mathrm{e}^{2\pi\nu}}\right)'\right)'\\
h_5(x) &= \int_0^\infty \mathrm{d}\nu\,\ln\left|1-\frac{\nu^2}{x^2}\right|\left(\frac{1}{\nu}\left(\frac{1}{\nu}\left(\frac{\nu^2}{1+\mathrm{e}^{2\pi\nu}}\right)'\right)'\right)'.
\end{aligned} \qquad (46)$$

We also see that $E_{as}^{(2)}$ has a pole of the form $\frac{1}{s}$ for $j = n = 1$. This pole will contribute to the heat-kernel coefficient A_2.

Now, since E_f contains no poles, we are able to write down the complete heat-kernel coefficients A_j, up to the order $j \leqslant 2$:

$$A_0 = 0 \qquad A_{1/2} = 0$$
$$A_1 = -4\pi R\alpha \qquad A_{3/2} = \pi^{3/2}\alpha^2 \qquad A_2 = -\frac{2}{3}\pi\frac{\alpha^3}{R}.$$

This coefficients are the same as in paper [5]. After performing the subtraction

$$(E_{as}^{(1)}+E_{as}^{(2)})-E^{div}$$

$E_{as}^{(1)}$ cancels completely and only $E_{as}^{(2)}$ (whitout its divergent portion) will contribute to the total energy. So we have, finally,

$$E_{as}|_{s=0} = -\frac{\alpha}{2\pi R}h_1(Rm)-\frac{\alpha^2}{8R^2m}h_2(Rm)+\frac{2\alpha^3+3\alpha}{48\pi R}h_3(Rm)$$
$$-\frac{\alpha}{8\pi R}h_4(Rm)+\frac{\alpha}{48\pi R}h_5(Rm). \qquad (47)$$

5. Asymptotics of E_{as}

As a test for the plots that we are going to do, we check analytically the behaviour of E_{as} for small and for large values of R. To do so we must find the corresponding asymptotics of the functions $h_n(x)$. For $R \to 0$ we find:

$$\begin{aligned}
&\lim_{R\to 0} h_1(Rm) \sim \ln(Rm) \cdot \tfrac{1}{48} + C_1 \\
&\lim_{R\to 0} h_2(Rm) \sim R^2 m^2 + C_2 \\
&\lim_{R\to 0} h_3(Rm) \sim -2\ln(Rm) \cdot (-\tfrac{1}{2}) + C_3 \\
&\lim_{R\to 0} h_4(Rm) \sim -2\ln(Rm) \cdot (-1) + C_4 \\
&\lim_{R\to 0} h_5(Rm) \sim -2\ln(Rm) \cdot (-4) + C_5
\end{aligned} \tag{48}$$

where the C_n are numbers resulting from the ν-integrations. Then we have

$$\lim_{R\to 0} E_{as} \sim -\frac{\alpha C_0}{16\pi R} - \frac{\alpha^3}{24\pi R}\left(\ln\frac{1}{Rm} - C_3\right) + \mathrm{O}(R^0) \tag{49}$$

where $C_0 = (-8C_1 + C_3 - C_4 + C_5/3) \sim 0.224$ and $C_3 \sim 1.96$.

For $R \to \infty$ we have

$$\begin{aligned}
&\lim_{R\to\infty} h_1(Rm) \sim -\frac{1}{m^2R^2}\frac{7}{1920} \\
&\lim_{R\to\infty} h_2(Rm) \sim \frac{1}{48} \\
&\lim_{R\to\infty} h_3(Rm) \sim \frac{1}{24m^2R^2} \\
&\lim_{R\to\infty} h_4(Rm) \sim -\frac{7}{480m^4R^4} \\
&\lim_{R\to\infty} h_5(Rm) \sim -\frac{16}{m^6R^6}\frac{31}{16\,128}
\end{aligned} \tag{50}$$

and we find

$$\lim_{R\to\infty} E_{as} \sim -\frac{1}{384}\frac{\alpha^2}{mR^2} + \mathrm{O}\left(\frac{1}{R^3}\right). \tag{51}$$

6. Numerical results

To numerically study the behaviour of E_{as} we rewrite it in the form

$$E_{as} = \frac{\alpha}{16\pi R} g_1(Rm) + \frac{\alpha^2}{16R} g_2(Rm) + \frac{\alpha^3}{16\pi R} g_3(Rm) \tag{52}$$

where the three functions $g_n(x)$ are given by

$$\begin{aligned}
g_1(x) &= -8h_1(x) + h_3(x) - 2h_4(x) + \tfrac{1}{3}h_5(x) \\
g_2(x) &= -2\frac{h_2(x)}{x} \\
g_3(x) &= \tfrac{2}{3}h_3(x).
\end{aligned} \tag{53}$$

As it is clear from (52), for small values of α E_{as} will behave like function $g_1(x)$. For large values of α, E_{as} will behave like function $g_3(x)$.

The plots of the three $g_n(x)$ functions are shown below in figure 1. In this, as in all the following plots, m is set equal to 1.

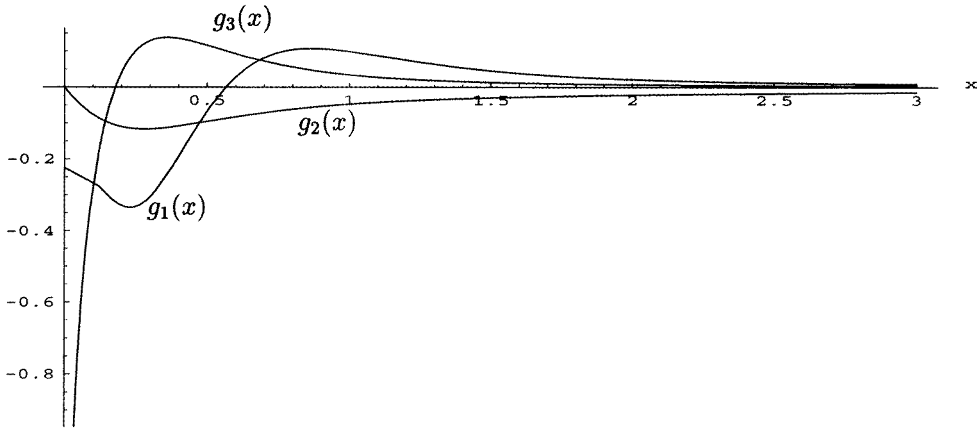

Figure 1. The curves of the $g_n(x)$ functions for a strength of the potential equal to 1.

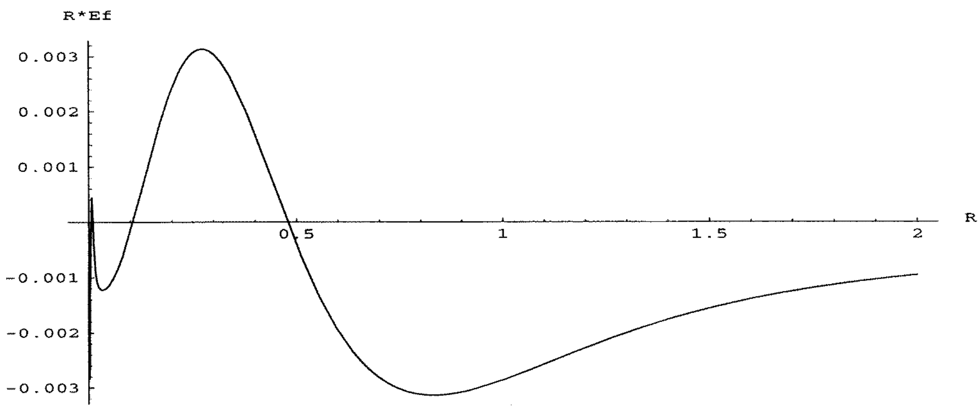

Figure 2. The curve of $R \times E_f(R)$ for a strength of the potential equal to 1. For $R = 0$ the curve converges to $R \times E_f \sim -3.9$.

For the complete quantum energy we still need the E_f contribution. In the expression (26) for E_f, after putting $s = 0$, we integrate by parts and obtain

$$E_f = \frac{1}{\pi} \sum_{l=0}^{\infty} \left(l + \frac{1}{2} \right) \int_m^{\infty} \frac{k}{\sqrt{k^2 - m^2}} (\ln f_\nu(\mathrm{i}k) - \ln f_\nu^{as}(\mathrm{i}k)) \, \mathrm{d}k. \tag{54}$$

This quantity cannot be further analytically simplified. Below we show (figure 2) a plot of $R \cdot E_f$ as function of R for $\alpha = 1$.

For the total GSE as a function of the radius of the shell we get the curves shown below (figures 3–5) for different values of the strength of the potential α.

7. Conclusion

We have obtained a representation of the renormalized GSE of a scalar massive field in the background of a semi-transparent shell containing convergent integrals of simple functions.

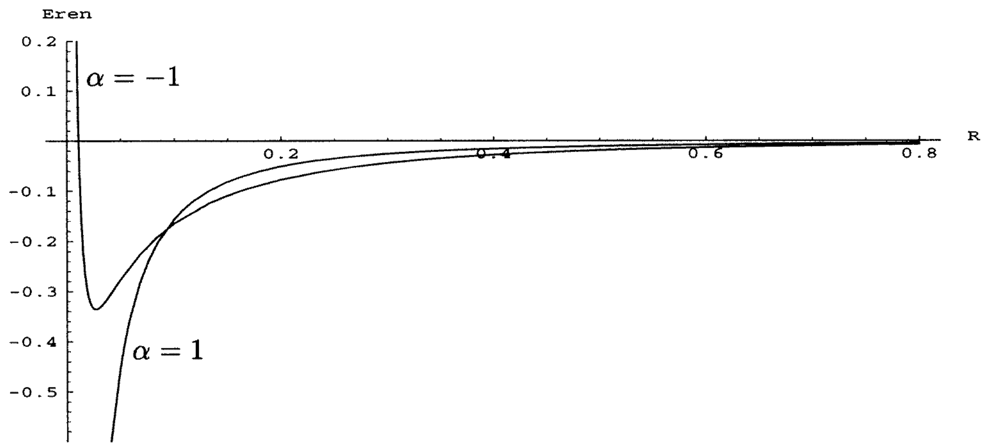

Figure 3. The renormalized vacuum energy $E_\varphi^{ren}(R)$ for positive and negative values of the potential.

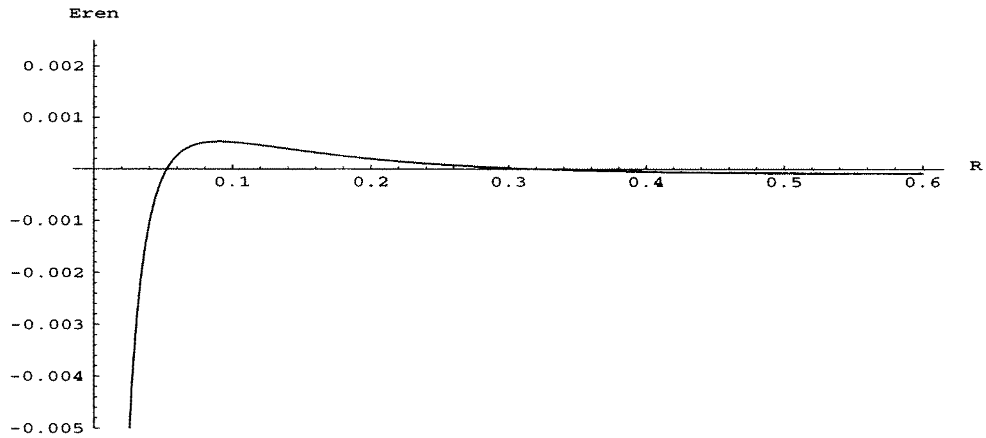

Figure 4. The renormalized vacuum energy $E_\varphi^{ren}(R)$ for $\alpha = 0.3$.

This expression is given by the sum of (52) and (54) and depends only on the two parameters of the classical system, namely the radius and the strength of the potential of the spherical shell. The plots of E_φ^{ren} as a function of the radius show that for repulsive potentials the renormalized GSE is positive only in some limited intervals of the radius axis and only when α is smaller than 1. For a strength of the potential larger than 1 the energy is always negative. This is the most striking conclusion of our work. For very large values of α the shell should be no more transparent and the problem should formally become a Dirichlet boundary condition problem. One could check this in equation (35) for the Jost function: here inserting a large α the addend 1 becomes negligible, then one would just have the product of the two modified Bessel I and K functions; such a Jost function is exactly the one for a perfectly reflecting spherical shell (Dirichlet boundary conditions). In that case the GSE is simply the sum of the energies inside

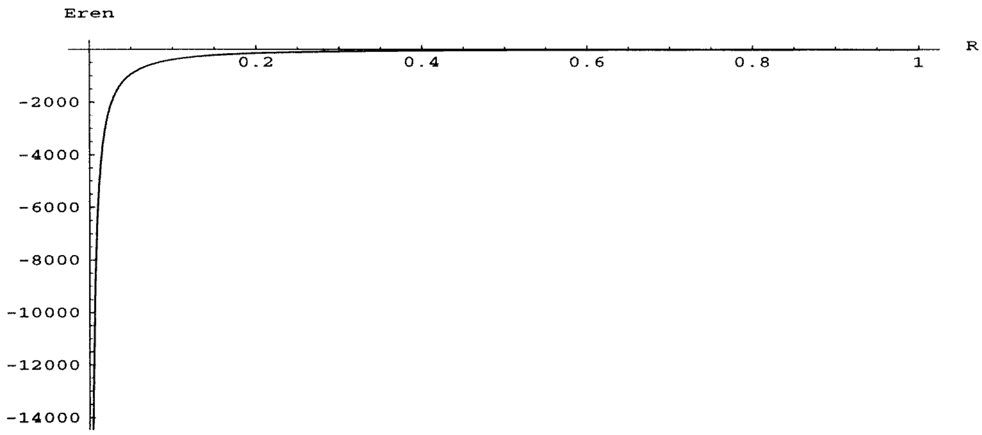

Figure 5. The renormalized vacuum energy $E_{\varphi}^{ren}(R)$ for $\alpha = 10$.

and outside the shell. Then formally we should have:

$$\lim_{\alpha\to+\infty} GSE^{semi-trans.} = GSE^{mirror}. \tag{55}$$

Now it is shown in [4] that the 'mirror', configuration, in the massive field case, always has positive GSE for repulsive potentials. This is in contradiction with our plots which show an opposite sign. Furthermore, the A_2 coefficient in paper [4] is zero. In our work, A_2 also remains a non-zero coefficient in the limit $\alpha \to +\infty$ demonstrating that the transition hypothesized in (55) is singular. For flat parallel semi-transparent boundaries with delta-function potential in the vacuum of a scalar massive field, the transition is actually fulfilled as shown in paper [16]. In the configuration analysed in this paper the limit (55) works only for the regularized GSE, as mentioned above, but after the renormalization the limit is no more valid. This means that the subtraction of the divergent part of the energy and the limit $\alpha \to +\infty$ are two non-commutative operations. We remark again that the results of this work cannot be directly applied to the case of a massless field, since the initial normalization condition would fail and the vacuum energy would not be univocally defined.

Acknowledgments

I thank M Bordag for advice and the referees for valuable remarks.

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